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Conclusion.—Each branch of engineering demands a somewhat different mathematical training from every other branch. For this reason every text book and every course must necessarily be a compromise. There are on the market some very acceptable books on algebra, trigonometry and the calculus. The pressing need is for equally acceptable books on analytic geometry, least squares and differential equations. There are, indeed, some very good American texts on differential equations, but, with possibly one exception, it does not appear that their authors have made any particular effort to find out which chapters are most useful to engineers, or to provide applications adapted to the needs of engineers, beyond certain standard types well known to all mathematicians.

A GEOMETRIC INTERPRETATION OF THE FUNCTION F IN HYPERBOLIC ORBITS, CORRESPONDING TO THAT OF THE ECCENTRIC ANOMALY E IN ELLIPTIC ORBITS.[1]

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The Geometric Interpretations of E and M in Elliptic Orbits.

If the elements of an elliptic orbit of a planet are given in the problem of the relative motion of two bodies, subject only to their mutual gravitation, the polar coördinates $r = SP$ (Fig. 1) and $v = \angle ASP$, the radius vector and true anomaly of the planet, can be computed for any time t in the following manner:

Compute the mean daily motion

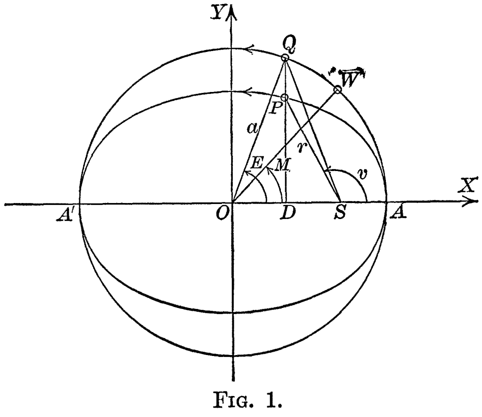

Fig. 1.

$$n = \frac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}},$$

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[1] This problem is given by F. R. Moulton in his *Introduction to Celestial Mechanics*, p. 159.

the mean anomaly $M = n(t - T)$, the eccentric anomaly E, by solving Kepler's equation

$$E - e \sin E = M,$$

and then $r = a(1 - e \cos E)$, and

$$\tan \frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}.$$

In these, a, T and e stand for the semi-major axis, time of perihelion passage and eccentricity of the relative orbit. The function E is first introduced into the problem analytically to aid in the integration of the equation of areas. It has the well known geometric meaning that it is always equal to the central angle AOQ, in the auxiliary circle, which corresponds to the true anomaly v.

We may give a geometric meaning to M by imagining a point W moving with uniform speed along the auxiliary circle in such a way that it would be at the perihelion point A of the orbit every time the planet was there. Thus M would be the central angle AOW described by the radius vector of this point.

It is further true that the area of the

$$\text{sector } AOQ = \tfrac{1}{2}a^2E,$$

and

$$\text{triangle } OSQ = \tfrac{1}{2}a \cdot ae \sin E.$$

Hence

$$\text{Area } ASQ = \tfrac{1}{2}a^2(E - e \sin E) = \tfrac{1}{2}a^2M$$

$$= \text{Area sector } AOW.$$

Thus M and E are proportional to the areas generated by SQ and OQ respectively, and would be numerically equal to twice those areas if $a = 1$.

THE GEOMETRIC INTERPRETATIONS OF F AND M IN HYPERBOLIC ORBITS.

The radius vector and true anomaly, r and v, in a hyperbolic orbit can be found in like manner for any instant t by the formulas,

$$(1) \qquad \left\{ \begin{aligned} n &= \frac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}}, \\ M &= n(t - T), \\ -F + e \sinh F &= M, \\ r &= a(-1 + e \cosh F), \\ \tan\frac{v}{2} &= \sqrt{\frac{e+1}{e-1}} \tanh \frac{F}{2}. \end{aligned} \right.$$

Like E, the function F is first introduced analytically as an auxiliary quantity.

We are to find geometric meanings for F and M analogous to those given to E and M in an elliptic orbit. For this purpose use the rectangular hyperbola having the same center and semi-major axis a as the orbit, and call it the auxiliary hyperbola. Also use the auxiliary circle with radius a and center at the center of the orbit. Thus in Fig. 2, $P'AP$ represents a hyperbolic orbit, with focus at S, perihelion at A, center at O, and P any position on the orbit; also $Q'AQ$ is the auxiliary hyperbola, with Q corresponding to P.

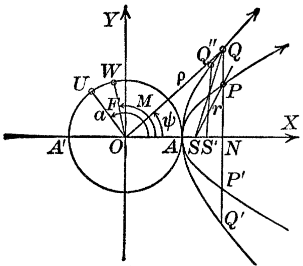

FIG. 2.

Join Q to O, and call $\angle NOQ = \psi$, $OQ = \rho$. Then the equation of the auxiliary hyperbola in rectangular coördinates is

$$x^2 - y^2 = a^2, \tag{2}$$

which expressed in polar coördinates gives $\rho^2(\cos^2\psi - \sin^2\psi) = a^2$, or

$$\rho^2 = \frac{a^2}{\cos 2\psi}. \tag{3}$$

Next compute the area bounded by OA, OQ and arc AQ.

$$OAQ = \tfrac{1}{2}\int_0^\psi \rho^2 d\psi = \frac{a^2}{2}\int_0^\psi \sec 2\psi d\psi = \frac{a^2}{4}\log\tan\left(\frac{\pi}{4}+\psi\right) = \frac{a^2}{2}K, \tag{4}$$

where we place

$$K = \tfrac{1}{2}\log\tan\left(\frac{\pi}{4}+\psi\right). \tag{5}$$

From this we readily deduce

$$\tan\psi = \tanh K. \tag{6}$$

Now the rectangular coördinates of Q may be expressed in terms of a parameter G by the relations

$$\begin{cases} x = ON = a\cosh G, \\ y = NQ = a\sinh G, \end{cases} \tag{7}$$

for these expressions satisfy (2). But

$$\tan\psi = \frac{NQ}{ON} = \frac{a\sinh G}{a\cosh G} = \tanh G. \tag{8}$$

Hence by (6)

$$K = G. \tag{9}$$

Again the area of the triangle

$$OSQ = \tfrac{1}{2}OS \cdot NQ = \tfrac{1}{2}ae \cdot a\sinh G. \tag{10}$$

Hence the area

$$ASQ = \tfrac{1}{2}a^2(e\sinh G - G). \tag{11}$$

Remembering that the changes in G are due to the motion of P, we have by comparison of (11) with (1)

$$G = F, \tag{12}$$

and

$$ASQ = \frac{a^2}{2} M; \tag{13}$$

also (4) becomes

$$OAQ = \frac{a^2}{2} F. \tag{14}$$

Thus M and F are proportional to the areas generated by SQ and OQ respectively, and would be numerically equal to twice those areas if $a = 1$.

But $M = n(t - T)$, that is, M is proportional to the time, from which it follows that as P describes its hyperbolic orbit, the corresponding point Q moves on the auxiliary hyperbola so that the law of areas holds with respect to the focus of the orbit.

From the form of the expressions (13) and (14) which represent the areas ASQ and OAQ, it is apparent that those areas are equivalent to circular sectors in a circle with radius a and central angles M and F respectively. We can thus imagine two points W and U traveling on the circle ARA', so related to the planet P that when P passes the perihelion point A, they will also pass through A. One of them, W, will go around the circle at a uniform speed, with its radius vector describing the area $\frac{a^2}{2} M =$ area ASQ. The other, U, will move around the circle at a non-uniform speed, with its radius vector OU describing the area $\frac{a^2}{2} F =$ area OAQ.

Now from (1)

$$\left\{ \begin{aligned} \frac{dM}{dt} &= \frac{k \sqrt{m_1 + m_2}}{a^{\frac{3}{2}}}, \\ \frac{dF}{dt} &= \frac{k \sqrt{m_1 + m_2}}{a^{\frac{3}{2}}(e \cdot \cosh F - 1)}, \end{aligned} \right. \tag{15}$$

which show that for $e < 2$ the non-uniformly moving point U would be moving faster than W at A; that they would be moving with equal speeds when

$$\cosh F = \frac{2}{e},$$

and thereafter the uniformly moving point W would move the faster.

By placing $F = M$ in (1) we find that W would overtake U at the time determined by

$$\sinh M = \frac{2M}{e}.$$

(15) also shows that the speed of the non-uniformly moving point would always decrease but not be zero. From (6) it follows, since $K = F$, that in an indefinitely great period of time this point U, as well as W, would make an indefinitely great number of revolutions about the circle.

The law of areas for hyperbolic motion gives the sector $ASP = \frac{a^2}{2}\sqrt{e^2-1}\,M$.

For a rectangular hyperbola $e = \sqrt{2}$ which substituted in this gives $\frac{a^2}{2}M$, or (13).

Hence it follows that the area described by SQ is equal to the area described in an equal time by the radius vector $S'Q''$ of a planet Q'' having the same mass and major axis as the planet P, but moving in an orbit which is a rectangular hyperbola.

THIRD CLEVELAND MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

As many readers of the MONTHLY may be unfamiliar with the history and the aims of the American Association for the Advancement of Science, the following data relating to these subjects may be of interest. In 1847 the Association of American Geologists and Naturalists agreed to resolve itself into the American Association for the Advancement of Science, and decided to hold the first meeting under the new management at Philadelphia in 1848.

At this first meeting of the Association its objects were stated to be as follows: "The objects of the Association are, by periodic and migratory meetings, to promote intercourse between those who are cultivating science in different parts of the United States, to give a stronger and more general impulse, and a more systematic direction to scientific research in our country, and to procure for the labors of scientific men increased facilities and a wider usefulness."

These objects have been modified only slightly during the sixty-four years since the Association was formed. For instance, the term United States has been replaced by the more general term America, and several of the meetings of the Association have been held in Canada. While most of the meetings have been in the east, the migrations have extended as far west as Denver and as far south as New Orleans. Four meetings have been held in each of the two cities Buffalo and Washington; and three in each of the four cities, Boston, New York, Philadelphia, and Cleveland. A few other cities have entertained the Association twice, but in most cases only a single visit to a given city has been made.

Similar organizations exist in many other countries. The British Association, organized in 1831, seems to occupy an especially prominent place in the scientific activity of that country and in the extent of its migrations. Its recent visits to South Africa and to Winnipeg, Canada, were marked with great liberality on